The values of $f''(o) = \alpha$ and $s'(o) = k$ as calculated from Eqs. (11–13) for different values of shock strength λ are compared in Table 1 with the numerical results of Mirels.

Table 1 Values of $f''(o)$ and $s'(o)$ for different λ

	$f''(o)$			$s'(o)$ for $\sigma = 0.72$	
λ	Eq. (12)	Eq. (11)	Mirels[6]	Eq. (13)	Mirels[6]
2	−1.0288	−1.0065	−1.0191	−0.8339	−0.8512
4	−4.2007	−4.0330	−4.0623	−1.0995	−1.1156
6	−8.4729	−8.0812	−8.1009	−1.3165	−1.3262

For vanishing velocity at the outer edge of the boundary layer or for large shock strengths (i.e., $\lambda \gg 1$), the equations for $f''(o)$ and $s'(o)$ given by Eickhoff are the same as Eqs. (11) and (13). For large values of σ, it can be shown that

$$s'(o) \approx -(2\lambda\sigma/\pi)^{1/2}$$

The Nusselt number (for $\sigma \to \infty$) is given by

$$Nu = -\frac{1}{(2\lambda)^{1/2}}(Re)^{1/2} s'(o) = \frac{1}{(\pi^{1/2})}(\sigma\, Re)^{1/2} \tag{14}$$

The preceding method can also be applied to the case when the wall is insulated. Following Mirels, the recovery factor $r(o)$ is expressed in quadrature form as

$$r(o) = \frac{2\sigma\alpha^2}{(\lambda-1)^2}\int_\eta^\infty e^{-\sigma F}\, d\xi \int_0^\xi e^{-(2-\sigma)F}\, d\theta \tag{15}$$

For $\sigma < 2$, the inner integral in Eq. (15) can be expressed as a sum of incomplete gamma functions when the series expansion (8) is substituted for the independent variable. If we retain only the first term in the series expansion (8) and, to be consistent with this approximation, use only the first term on the RHS of Eq. (9) to evaluate α, then Eq. (15) integrates to

$$r(o) = \frac{4}{\pi}\left(\frac{\sigma}{2-\sigma}\right)^{1/2} \sin^{-1}\left(\frac{2-\sigma}{2}\right)^{1/2} \tag{16}$$

Thus, to the first order, the recovery factor does not depend on the shock strength at all; instead it depends only on the Prandtl number. For $\sigma = 0.72$, $r(o) = 0.8855$. It may be noted here that Eq. (16) is the same as that given by Emmons[9] for the recovery factor of an accelerated plate starting from rest. The result for $r(o)$ can be improved by higher approximations and this has been done in Refs. 4 and 5 by retaining the first two terms in the series expansion for F. However, $r(o)$ was found to have a weak dependence on the shock strength. Numerical results of Mirels also show a similar weak dependence.

As mentioned by Eickhoff, these analytical solutions show general tendencies. Also, the simplicity of the solution obtained by one-term approximation suggests the following empirical relations for α and k:

$$\alpha = -(0.211 + 0.575\lambda^{1/2})(\lambda - 1) \tag{17}$$

and

$$k = -(0.202 + 0.459\lambda^{1/2}) \quad \text{for} \quad \sigma = 0.72 \tag{18}$$

The results presented here have the advantage of being applicable over a wide range of values of λ and σ.

After determining the wall derivatives $f''(o)$ and $s'(o)$ which are of major interest, it is possible to calculate the velocity and temperature profiles.[4] The calculations indicate that the free-stream conditions are approached earlier than in the exact solutions.

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Technical Comments

Comment on "Criteria for Selecting Curves for Fitting to Data"

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I. Introduction

IN an Aug. 1970 paper,[1] T. J. Dylewski proposed an interesting criterion for discriminating between "oversmoothing" and "undersmoothing" in the least squares fitting of equations to data. This criterion is given by

$$B = S'/S''$$

where S' is the sum of squared deviations from fitting a given curve to all the data and S'' is the pooled sum of squared deviations from fitting a curve of the same type to each of two halves of the data (for simplicity, let the sample size, N, be an even integer). Among the advantages the author cites for this criterion is that it does not permit tests of significance. Before showing that it actually does, and indeed that an evaluation of significance is a necessity, it may be well to comment on statistical tests.

II. Statistical Tests

The author uses the term, test of significance, to describe the process of choosing a Type I Error risk (level of significance), calculating a test statistic and then either accepting or rejecting the hypothesis being considered. There has been some effort, though not entirely successful, in the statistical literature to refer to this procedure as hypothesis testing, or more explicitly, accept/reject decision-making. While hypothesis testing seems to have direct applicability to acceptance sampling, where indeed one is required to make a series of accept/reject decisions, and perhaps to certain confirmatory experiments, it is of questionable value in

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exploratory investigations and data analyses, such as where the need for curve fitting often occurs. The statistical analysis in these cases is generally referred to as significance testing, or significance evaluation, and, in contrast to the author's use of the term, this refers to the process of calculating a significance level from the data and reporting this as a summary statistic which measures the consonance of the data with some hypothesized model. Subsequent decisions may be based at least in part on observed significance levels, but the point is that one is not required beforehand to specify all hypotheses and risk levels. To do so is contrary to the nature of scientific investigation and, as the author pointed out, often involves an unsatisfactory element of arbitrariness. Much statistical literature leaves the unfortunate impression that the number one problem facing statisticians is the choice of the Type I error level, but that it has been solved for them since time immemorial by taking this risk to be 0.05 or 0.01. Those texts which mention significance tests and contrast them with tests of hypotheses, often do so with a note of apology for their lack of mathematical rigor. This is unfortunate. Users of statistics need to be aware that statistics can be used for learning from data as well as for making decisions which have a given probability of success under certain idealized conditions. A more extensive discussion of this matter is given elsewhere.[2–4]

III. The Distribution of B

The quantity B is a statistic, a function of data. For it to be of use, it is necessary to have some scale of measurement to judge it against. Only then can we tell how large a large B value is, or how much larger $B = 9.19$ is than $B = 1.58$. We now consider a statistical scale of measurement, which is the probability distribution of B under sampling from a given model.

B is always at least 1.0, so we can write it as

$$B = S'/S'' = 1 + R'/S''$$

where $R' \geqq 0$. Consider the polynomial models

$$Y_1 = \beta_{10} + \beta_{11}X + \beta_{12}X^2 + \ldots + \beta_{1K}X^K + e$$
$$Y_2 = \beta_{20} + \beta_{21}X + \beta_{22}X^2 + \ldots + \beta_{2K}X^K + e$$

where the subscripts, 1 and 2, refer to the two halves of the data, and the hypothesis

$$\beta_{10} = \beta_{20}; \quad \beta_{11} = \beta_{21}; \ldots; \quad \beta_{1K} = \beta_{2K}$$

that is, consider the case, which one might label minimum bias, where the two-data halves are generated by the same model. In this case, it can be shown (see Graybill[5]) that

$$F = \frac{R'/(K+1)}{S''/(N-2K-2)} = \frac{(B-1)(N-2K-2)}{(K+1)}$$

has an F distribution with $K+1$ and $N-2K-2$ degrees of freedom. Departures from this hypothesis tend to inflate R', and hence B, so one can quantify the comparison of observed data to this model by calculating

$$P = \text{Prob}[F(K+1, N-2K-2) \geqq \text{observed } F]$$

that is by making a test of significance of the closeness of the data to the minimum bias hypothesis. [$F(\cdot,\cdot)$ is an F random variable with the indicated degrees of freedom.] The larger P is, the closer, or more in agreement, the data are to the hypothesis. Note that this criterion takes into consideration the sample size and the degree of the polynomial, neither of which enter into the author's comparison of B-values. The simplest case, $K = 0$, is just the usual test for equality of two means and realizing this helps clarify just what is being measured by B.

IV. Example

Consider the illustrative example of the author and his Figs. 4–8. Table 1 gives B, F, the degrees of freedom, and P for polynomials of degree 0–4. Note that the minimum B in this case agrees with the maximum P. However, both the second and fourth degree polynomials, with $B = 1.95$ and 9.19, respectively, yield larger P-values than the linear fit which has $B = 1.58$. That is, the B's for the second and fourth degree curves are not as unusual under the minimum bias model as the smaller B for the linear fit. Thus, comparing B-values as though they were points on the real line could be quite misleading.

Table 1 Analysis of sample data

Degree	B	F	Degrees of freedom	P
0	6.31	53.1	(1, 10)	$\approx 10^{-5}$
1	1.58	2.33	(2, 8)	0.16
2	1.95	1.90	(3, 6)	0.23
3	1.43	0.43	(4, 4)	0.78
4	9.19	3.28	(5, 2)	0.25

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Reply by Author to R. G. Easterling

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THE part of the Comment headed "Statistical Tests" expresses generalized personal attitudes towards uses of statistical tests and their nomenclature. This is a popular debating topic for some statisticians who, nevertheless, often seem to carry out their professional activities not much differently or more ably than their nominal opponents. This material has only incidental pertinence to my paper.

It can be inferred from the rest of the comment that the paper was not carefully read. I did not, for example, state that the bias-ratio, B, does not permit a test of significance; I did say that such a test is not available. I did not claim the lack of a test of significance as an advantage; I did say that the bias-ratio criterion has the advantage of not requiring the selection of a significance level.

More importantly, the comment presents an erroneous derivation of the distribution of B in an attempt to calculate its significance. This is done by an unjustifiable transformation of the B-distribution into the F-distribution.

I had investigated the possibility of using the variance-ratio test based on the F-distribution prior to devising the bias-ratio criterion. Some of the reasons for discarding the former are stated in the paper. Since the commenter appears to have overlooked them, I will restate them.

The F-distribution is applicable to normally-distributed random deviations. This condition is usually realized adequately closely only when a function without fitting bias is found. For any other function, howsoever chosen for "best-fit," the deviations may be far from normal and random. Moreover, the F-distribution applies to one test only, not to a sequence of tests made for the purpose of locating an extreme probability. Such

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